

ON REPRESENTATIONS BY EGYPTIAN FRACTIONS

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To Lucian Bădescu on the occasion of his seventieth birthday

ABSTRACT. We bound the entries of the representations of a rational number as a sum of Egyptian fractions.

INTRODUCTION

Let (X, B) be a log canonical model with standard coefficients. That is (X, B) is a log variety with log canonical singularities, $K_X + B$ is \mathbb{Q} -ample, and the coefficients of B belong to the standard set $\{1 - \frac{1}{m}; m \in \mathbb{Z}_{\geq 1}\} \cup \{1\}$. The volume of (X, B) is $v = \sqrt[d]{(K_X + B)^d}$, where $d = \dim X$. By [1, 6, 3, 4], the volume v belongs to a DCC set, and there exists a positive integer r , bounded above only in terms of d and v , such that the linear system $|r(K_X + B)|$ is base point free (in particular, $r(K_X + B)$ is a Cartier divisor). The DCC property means that if t is a real number and $v > t$, then $v \geq t + \epsilon$, where ϵ depends only on d and t .

In this note we estimate the gap and index bounds ϵ and r in the simplest possible case, when X is a projective space and the components of B are hyperplanes in general position. According to [6], the sharp bounds of the simplest case are possibly optimal in the general case.

To formulate our main result, we define a sequence of integers $(u_{p,q})_{p,q \geq 1}$ by the recursion $u_{1,q} = q$, $u_{p+1,q} = u_{p,q}(u_{p,q} + 1)$. Then $u_{p,q}$ is a polynomial in q with leading term $q^{2^{p-1}}$, and the following formulas hold:

$$\sum_{i=1}^p \frac{1}{1 + u_{i,q}} = \frac{1}{q} - \frac{1}{u_{p+1,q}}, \quad \prod_{i=1}^p (1 + u_{i,q}) = \frac{u_{p+1,q}}{q}.$$

The sequence $(1 + u_{p,1})_{p \geq 1} = (2, 3, 7, 43, \dots)$ is called the Sylvester sequence in the literature (see [5, 6]), and also the sequence $t_{p,q} = 1 + u_{p,q}$ was considered in [7].

Theorem 0.1. *Let $(\mathbb{P}^d, \sum_i b_i E_i)$ be a log structure such that the $(E_i)_i$ are general hyperplanes and the coefficients b_i belong to the standard set. Let $v = \deg(K + B)$. Let $t \geq 0$ be a rational number, with $qt \in \mathbb{Z}$ for some integer $q \geq 1$.*

- a) *If $v > t$, then $v \geq t + \frac{q(1-\{t\})}{u_{\lfloor t \rfloor + d + 3, q}}$.*
- b) *If $v = t$, then there exists an integer $1 \leq r \leq \frac{u_{\lfloor t \rfloor + d + 2, q}}{q(1-\{t\})}$ such that the linear system $|r(K + B)|$ is base point free.*

Theorem 0.1 is in fact combinatorial, about bounding the representations of a given rational number as a sum of Egyptian fractions. Any positive rational number x admits

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a representation as a sum of Egyptian fractions

$$x = \frac{1}{m_1} + \cdots + \frac{1}{m_k},$$

where m_i are positive integers and k is sufficiently large. If $x = \frac{p}{q}$ is the reduced form, we can write $x = \sum_{i=1}^p \frac{1}{q}$. From a representation with k terms we can construct another one with $k+1$ terms, using the formula

$$\frac{1}{m} = \frac{1}{m+1} + \frac{1}{m(m+1)}.$$

A canonical representation is provided by the *greedy algorithm*: if $x > 0$, let $m \geq 1$ be the smallest integer such that $mx \geq 1$, and replace x by $x - \frac{1}{m}$; if $x = 0$, stop. After each step, the numerator of the reduced fraction decreases strictly, and therefore the algorithm stops in finite time, and produces a representation of x as a sum of k Egyptian fractions ($k \leq \lfloor x \rfloor + q\{x\}$ if $qx \in \mathbb{Z}$).

If k is fixed, it is easy to see that x admits only finitely many representations with k Egyptian fractions. The following is an effective version of this fact, which is a restatement of Theorem 0.1.

Theorem 0.2. *Let $1 \leq m_1 \leq \cdots \leq m_k$ be integers. Let $\delta \geq -1$ with $q\delta \in \mathbb{Z}$ for some integer $q \geq 1$.*

- a) *If $\sum_{i=1}^k \frac{1}{m_i} < k - \delta$, then $\sum_{i=1}^k \frac{1}{m_i} \leq k - \delta - \frac{q(1-\{\delta\})}{u_{\lfloor \delta \rfloor + 2, q}$.*
- b) *If $\sum_{i=1}^k \frac{1}{m_i} = k - \delta$, then $\text{lcm}(m_1, \dots, m_k) \leq \frac{u_{\lfloor \delta \rfloor + 1, q}}{q(1-\{\delta\})}$.*

Moreover, equality holds in a) if and only if $\delta < 0$, or $\delta = \frac{r}{q} \in [0, 1)$, $(m_i)_i = (1, \dots, 1, \frac{1+q}{r})$, or $1 \leq \delta = s - \frac{1}{q}$, $(m_i)_i = (1, \dots, 1, 1 + u_{1,q}, \dots, 1 + u_{s,q})$. And equality holds in b) if and only if $\delta = s - \frac{1}{q}$, $(m_i)_i = (1, \dots, 1, 1 + u_{1,q}, \dots, 1 + u_{s,q}, u_{s+1,q})$, or $\delta = 2 - \frac{r}{q}$ and $(m_i)_i = (1, \dots, 1, \frac{1+q}{r}, \frac{q(1+q)}{r})$.

The case $k - \delta = 1$ is known (Kellogg [5], Curtiss [2], Soundararajan [8]), with b) replaced by the same bound for m_k instead of the least common multiple. We use the method of Soundararajan [8].

1. PROOF OF ESTIMATES

Lemma 1.1 ([8]). *Consider real numbers $x_1 \geq x_2 \geq \cdots \geq x_n > 0$ and $y_1 \geq y_2 \geq \cdots \geq y_n > 0$ such that $\prod_{i \leq k} x_i \geq \prod_{i \leq k} y_i$ for all k . Then $\sum_i x_i \geq \sum_i y_i$, with equality if and only if $x_i = y_i$ for all i .*

Proof. Soundararajan [8] deduces this lemma from Muirhead's inequality. We give here a direct proof, by induction on n . If $x_i = y_i$ for some i , we may remove the i -th terms from both n -tuples, and conclude by induction; therefore we may suppose $x_i \neq y_i$ for every i . If $x_i > y_i$ for all i , the conclusion is clear. Suppose $x_i < y_i$ for some i . Let $l = \min\{i; x_i < y_i\}$. Then $l > 1$ and $x_i > y_i$ for every $i < l$. Let $t = \min\{\frac{x_{l-1}}{y_{l-1}}, \frac{y_l}{x_l}\} > 1$. Define $(x'_i)_i$ by $x'_i = x_i$, for $i \notin \{l-1, l\}$, and $x'_{l-1} = \frac{x_{l-1}}{t}$, $x'_l = tx_l$. One checks that $x'_1 \geq x'_2 \geq \cdots \geq x'_n > 0$, $\prod_{i=1}^k x'_i \geq \prod_{i=1}^k x_i$ for all k , and $x_{l-1} + x_l > x'_{l-1} + x'_l$, hence $\sum_{i=1}^n x_i > \sum_{i=1}^n x'_i$. Since either $x'_{l-1} = y_{l-1}$ or $x'_l = y_l$, $\sum_{i=1}^n x'_i \geq \sum_{i=1}^n y_i$ by induction. Therefore $\sum_{i=1}^n x_i > \sum_{i=1}^n y_i$. The claim on equality is clear. \square

Lemma 1.2. *Consider real numbers $x_1 \geq x_2 \geq \dots \geq x_n > 0$ and $y_1 \geq y_2 \geq \dots \geq y_n > 0$ such that $\sum_{i \geq k} x_i \geq \sum_{i \geq k} y_i$ for all k . Then $\prod_i x_i \geq \prod_i y_i$, with equality if and only if $x_i = y_i$ for all i .*

Proof. As in the previous lemma we use induction on n , so that we may suppose $x_i \neq y_i$ for every i . In particular, $x_n > y_n$. If $x_i > y_i$ for every i , the claim is clear. So suppose that $x_i < y_i$ for some i . Let $k = \max\{i; x_i < y_i\}$. Then $k < n$ and $x_i > y_i$ for every $i \geq k+1$. In particular,

$$y_{k+1} < x_{k+1} \leq x_k < y_k.$$

Define $(y'_i)_i$ by $y'_i = y_i$ for $i \notin \{k, k+1\}$, $y'_k = y_k - \epsilon$, $y'_{k+1} = y_{k+1} + \epsilon$, where $\epsilon = \min\{x_{k+1} - y_{k+1}, y_k - x_k\} > 0$. The following hold:

- $y'_1 \geq \dots \geq y'_n > 0$.
- $\sum_{i \geq j} y_i \leq \sum_{i \geq j} y'_i$, with equality for $j \neq k+1$. And $\sum_{i \geq j} x_i \geq \sum_{i \geq j} y'_i$ for all j .
- $y'_k y'_{k+1} - y_k y_{k+1} = \epsilon(y_k - y_{k+1} - \epsilon) > 0$. Therefore $\prod_i y'_i > \prod_i y_i$.

By induction, the claim holds for (x_i) and (y'_i) , since either $x_k = y'_k$ or $x_{k+1} = y'_{k+1}$. Therefore $\prod_i x_i \geq \prod_i y'_i$, so that $\prod_i x_i > \prod_i y_i$. \square

For the next proposition we need the following lemma whose proof is obvious.

Lemma 1.3. *Let n, p, q be positive integers with $1 - \frac{1}{n} \leq \frac{p}{q} < 1$. Then $n \leq q$.*

Proposition 1.4. *Let $s \geq 0, 1 \leq r \leq q$ be integers. If $1 \leq n_1 \leq \dots \leq n_k$ are integers such that $\sum_{i=1}^k \frac{1}{n_i} < k - s + \frac{r}{q}$, then $\sum_{i=1}^k \frac{1}{n_i} \leq k - s + \frac{r}{q} - \frac{r}{u_{s+1,q}}$. Equality holds if and only if $n_i = 1$ for $i \leq k - s$ and $n_i = \frac{1+u_{i-k+s,q}}{r}$ for $i > k - s$.*

Proof. We use induction on s to prove that if $1 \leq n_1 \leq \dots \leq n_k$ are integers such that $k - s + \frac{r}{q} - \frac{r}{u_{s+1,q}} \leq \sum_{i=1}^k \frac{1}{n_i} < k - s + \frac{r}{q}$, then $n_i = 1$ for $i \leq k - s$ and $n_i = \frac{1+u_{i-k+s,q}}{r}$ for $i > k - s$.

If $s = 0$, then $k \leq \sum_{i=1}^k \frac{1}{n_i} < k + \frac{r}{q}$, so that $n_i = 1$ for all i .

Let $s \geq 1$. The right inequality yields $s \leq k$. Denote $m_i = 1$ for $1 \leq i \leq k - s$ and $m_i = \frac{1+u_{i-k+s,q}}{r}$ for $k - s < i \leq k$. We have

$$\sum_{i=1}^k \frac{1}{m_i} = k - s + \frac{r}{q} - \frac{r}{u_{s+1,q}}, \quad \prod_{i=1}^k m_i = \frac{u_{s+1,q}}{r^s q}.$$

Our hypothesis can be rewritten as

$$1 - \frac{q}{u_{s+1,q}} \leq \frac{q}{r} \left(s - k + \sum_{i=1}^k \frac{1}{n_i} \right) < 1.$$

The middle term can be represented as a fraction with denominator $r \prod_i n_i$. By Lemma 1.3, $\frac{u_{s+1,q}}{q} \leq r \prod_{i=1}^k n_i$. Therefore $\prod_{i=1}^k m_i \leq \prod_{i=1}^k n_i$. Then we can define

$$j = \max\{1 \leq l \leq k; \prod_{i \geq l} m_i \leq \prod_{i \geq l} n_i\}.$$

Assume $j = k$, that is $m_k \leq n_k$. Then $\sum_{i=1}^{k-1} \frac{1}{m_i} \leq \sum_{i=1}^{k-1} \frac{1}{n_i} < (k-1) - (s-1) + \frac{r}{q}$. By induction, $n_i = m_i$ for every $i \leq k-1$. It follows that $n_k = m_k$.

Assuming $j < k$, we derive a contradiction. Then $\prod_{i \geq j} n_i \geq \prod_{i \geq j} m_i$ and $\prod_{i \geq p} n_i < \prod_{i \geq p} m_i$ for every $j < p \leq k$. It follows that $\prod_{i=j}^p n_i > \prod_{i=j}^p m_i$ for every $j \leq p < k$. We rewrite this as

$$\prod_{i=j}^p \frac{1}{m_i} \geq \prod_{i=j}^p \frac{1}{n_i} \quad (j \leq p \leq k),$$

with strict inequality for $p \neq k$. By Lemma 1.1, $\sum_{i=j}^k \frac{1}{m_i} > \sum_{i=j}^k \frac{1}{n_i}$. On the other hand, $\sum_{i=1}^{j-1} \frac{1}{n_i} < k - s + \frac{r}{q}$. By induction, $\sum_{i=1}^{j-1} \frac{1}{n_i} \leq \sum_{i=1}^{j-1} \frac{1}{m_i}$. Therefore $\sum_{i=1}^k \frac{1}{n_i} < \sum_{i=1}^k \frac{1}{m_i}$, a contradiction. \square

Remark 1.5. Notice that since $1 + u_{1,q}$ and $1 + u_{2,q}$ are relatively prime, if $s \geq 2$ equality is achieved only for $r = 1$.

Proposition 1.6. *Let $s \geq 0$ and $1 \leq r \leq q$ be integers. If $1 \leq n_2 \leq \dots \leq n_k$ are integers such that $\sum_{i=1}^k \frac{1}{n_i} = k - s + \frac{r}{q}$, then $\text{lcm}(n_1, \dots, n_k) \leq \frac{u_{s,q}}{r}$. Equality holds if and only if $n_i = 1$ for $1 \leq i \leq k - s$, $n_i = \frac{1+u_{i-k+s,q}}{r}$ for $k - s < i < k$ and $n_k = \frac{u_{s,q}}{r}$.*

Proof. We prove by induction on s that if $1 \leq n_2 \leq \dots \leq n_k$ are integers such that $\sum_{i=1}^k \frac{1}{n_i} = k - s + \frac{r}{q}$ and $\text{lcm}(n_1, \dots, n_k) \geq \frac{u_{s,q}}{r}$, then $n_i = 1$ for $1 \leq i \leq k - s$, $n_i = \frac{1+u_{i-k+s,q}}{r}$ for $k - s < i < k$ and $n_k = \frac{u_{s,q}}{r}$.

It follows that $s \geq 1$. If $s = 1$, we must have $(n_i) = (1, \dots, 1, \frac{q}{r})$, so the conclusion holds. Suppose $s \geq 2$. Let $m_i = 1$ for $1 \leq i \leq k - s$, $m_i = \frac{1+u_{i-k+s,q}}{r}$ for $k - s < i < k$ and $m_k = \frac{u_{s,q}}{r}$. We have $m_1 \leq \dots \leq m_k$, $\sum_{i=1}^k \frac{1}{m_i} = k - s + \frac{r}{q}$ and $\prod_{i=1}^k m_i = \frac{u_{s,q}^2}{r^s q}$.

Step 1: We claim that $\sum_{i \geq l} \frac{1}{n_i} \geq \sum_{i \geq l} \frac{1}{m_i}$ for every $1 \leq l \leq k$.

Indeed, equality holds for $l = 1$. Let $1 < l \leq k - s + 1$. Then $\sum_{i < l} \frac{1}{n_i} \leq l - 1 = \sum_{i < l} \frac{1}{m_i}$. Therefore $\sum_{i \geq l} \frac{1}{n_i} \geq \sum_{i \geq l} \frac{1}{m_i}$. Let $k - s + 1 < l \leq k$. Then $\sum_{i=1}^{l-1} \frac{1}{n_i} < k - s + \frac{r}{q} = (l - 1) - (l + s - k - 1) + \frac{r}{q}$. By Proposition 1.4, $\sum_{i=1}^{l-1} \frac{1}{n_i} \leq \sum_{i=1}^{l-1} \frac{1}{m_i}$. Therefore $\sum_{i \geq l} \frac{1}{n_i} \geq \sum_{i \geq l} \frac{1}{m_i}$.

Step 2: By Step 1 and Lemma 1.2, we obtain $\prod_i \frac{1}{n_i} \geq \prod_i \frac{1}{m_i}$, that is $\prod_i n_i \leq \prod_i m_i$. And equality holds if and only if $n_i = m_i$ for all i .

Step 3: Denote $L = \text{lcm}(n_1, \dots, n_k)$. We claim that $L^2 \leq q \prod_{i=1}^k n_i$.

Indeed, $q \mid L$ and $n_i \mid \text{lcm}(q, n_1, \dots, \widehat{n_i}, \dots, n_k)$ for all i . Fix a prime p . The power of p in L is the highest power of p occurring in the prime decomposition of q, n_1, \dots, n_k . From above, the maximum is attained at least twice. Therefore $L^2 \leq q \prod_{i=1}^k n_i$.

Step 4: We obtain $L^2 \leq q \prod_{i=1}^k n_i \leq q \prod_{i=1}^k m_i = \frac{u_{s,q}^2}{r^s}$. Since $s \geq 2$, we obtain $L \leq \frac{u_{s,q}}{r}$. We assumed the opposite inequality, so $L = \frac{u_{s,q}}{r}$. It follows that $\prod_{i=1}^k n_i = \prod_{i=1}^k m_i$, so $n_i = m_i$ for all i . \square

Remark 1.7. Note that equality is achieved if $\frac{1+u_{i-k+s,q}}{r}$ for $k - s < i < k$ and $\frac{u_{s,q}}{r}$ are integers, that is if and only if $s = 1$ and $r \mid q$, or $s = 2$ and $r \mid 1 + q$, or $s \geq 3$ and $r = 1$.

Proof of Theorem 0.2. Write $\delta = s - \frac{r}{q}$, where $s = \lfloor \delta \rfloor + 1$ and $r = q(1 - \{\delta\})$. Then $k - \delta = k - s + \frac{r}{q}$, and we may apply Propositions 1.4 and 1.6. \square

Proof of Theorem 0.1. Order the coefficients of B as $0 \leq b_1 \leq \dots \leq b_k < 1 = b_{k+1} = \dots = b_{k+c}$. Let $b_i = 1 - \frac{1}{m_i}$, for $1 \leq i \leq k$. Then $\sum_{i=1}^k \frac{1}{m_i} = k - (d - c + 1 + v)$. Denote $r = \text{lcm}(m_i)$. By Theorem 0.2, $r \leq \frac{u_{\lfloor t \rfloor + d - c + 2, q}}{q(1 - \{t\})}$. Then rB is a divisor with integer

coefficients, and since the ambient space is \mathbb{P}^d , the semipositive Cartier divisor $r(K + B)$ is base point free. \square

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